Lecture 7

Spring Rates (Sections 5-1, 5-16 and 5-11 to 5-15)

Relationship between force and deflection,

$$F = f(y) \tag{1}$$

the spring rate is defined as,

$$k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy} \tag{2}$$

Most spring rate problems are linear and therefore,

$$k = \frac{F}{y} \tag{3}$$

where k is termed the spring rate constant.

For straight bar loaded axially,

$$\delta = \frac{Fl}{AE} \tag{4}$$

where

$$k = \frac{AE}{l} \tag{5}$$

For the angular deflection of a round bar

$$\theta = \frac{Tl}{GJ} \tag{6}$$

The torsional spring rate is

$$k = \frac{T}{\theta} = \frac{GJ}{l} \tag{7}$$

For beam deflection

$$y = f(x) \tag{8}$$

$$\theta = \frac{dy}{dx} \tag{9}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \tag{10}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3} \tag{11}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4} \tag{12}$$

and we use either singularity functions or numerical integration to solve these for the deflections. Most engineers use tables (ie, Roark and Young).

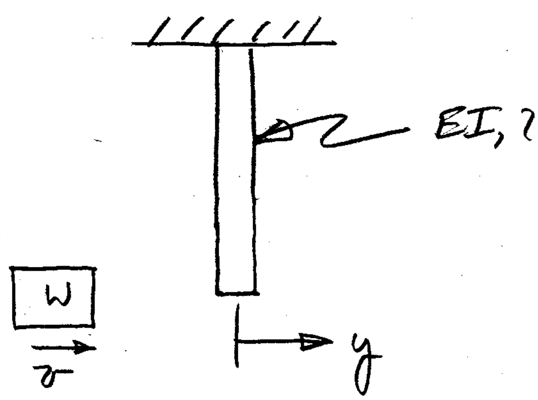
Shock and Impact

impact - the collision of two masses with initial relative velocities.

shock - any suddenly applied force or disturbance

Example (Simple Case)

Weight W is moving at a velocity of v when is strikes a cantilever beam. What is the maximum deflection?



the spring rate for a cantilever beam is

$$k = \frac{F}{y} = \frac{3EI}{l^3} \tag{13}$$

If we start counting time at the moment the weight strikes the beam, ($t=0$, $y=0$ and $\dot{y}=v$) the differential equation is,

$$\frac{W}{g}\ddot{y} = -ky \tag{14}$$

The solution to the equation is,

$$y = A\cos\left(\frac{kg}{W}\right)^{\frac{1}{2}}t + B\sin\left(\frac{kg}{W}\right)^{\frac{1}{2}}t \tag{15}$$

and the velocity is,

$$\dot{y} = -A\left(\frac{kg}{W}\right)^{\frac{1}{2}}\sin\left(\frac{kg}{W}\right)^{\frac{1}{2}}t + B\left(\frac{kg}{W}\right)^{\frac{1}{2}}\cos\left(\frac{kg}{W}\right)^{\frac{1}{2}}t \tag{16}$$

Using the initial conditions,

$$A = 0 \tag{17}$$

$$B = \frac{v}{\left(\frac{kg}{W}\right)^{\frac{1}{2}}} \tag{18}$$

the solution to the equation is,

$$y = \frac{v}{\left(\frac{kg}{W}\right)^{\frac{1}{2}}}\sin\left(\frac{kg}{W}\right)^{\frac{1}{2}}t \tag{19}$$

This equation is only valid when the mass is in contact with the cantilever beam. The maximum deflection is,

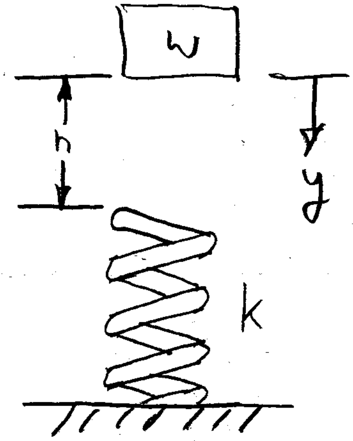
$$y_{max} = \frac{v}{\left(\frac{kg}{W}\right)^{\frac{1}{2}}} = v\left(\frac{Wl^3}{3EIg}\right)^{\frac{1}{2}} \tag{20}$$

and the maximum bending moment becomes,

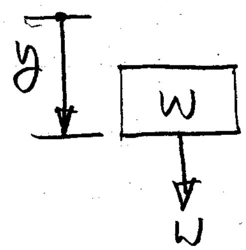
$$M_{max} = kl\,y_{max} = v\left(\frac{3EIW}{gl}\right)^{\frac{1}{2}} \tag{21}$$

<u>Example (More Difficult)</u>

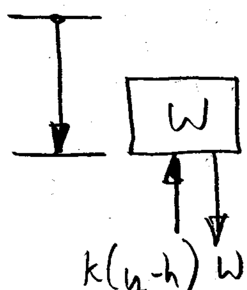
Consider a weight W falling a distance h and impacting a structure with a spring rate of k. Write equations which describe the motion.



Two free-body diagrams are required, one for $y \le h$



and the other for $y > h$,



Summing the forces we have,

$$\frac{W}{g}\ddot{y} = W \qquad y \le h \tag{22}$$

$$\frac{W}{g}\ddot{y} = -k(y-h) + W \qquad y > h \tag{23}$$

these equations are termed "piecewise differential equations."

The solution to eq. 22 is,

$$y = \frac{gt^2}{2} \qquad y \le h \tag{24}$$

the time at which the weight strikes the spring $(y = h)$ is,

$$t_1 = \left(\frac{2h}{g}\right)^{\frac{1}{2}} \tag{25}$$

and the velocity at t_1 is,

$$\mathring{y} = gt \qquad (y \le h) \tag{26}$$

and at t_1,

$$\mathring{y}_1 = gt_1 = (2gh)^{\frac{1}{2}} \quad (y > h) \tag{27}$$

To solve eq. 23, we first define $t' = t - t_1$. From classical methods we find,

$$y = \left[\left(\frac{W}{k}\right)^2 + \frac{2Wh}{k}\right]^{\frac{1}{2}} \cos\left[\left(\frac{kg}{W}\right)t' - \phi\right] + h + \frac{W}{k} \quad (y > h) \tag{28}$$

where ϕ is the phase angle (generally of no interest to us).

The maximum deflection of the spring becomes,

$$\delta = y - h = \frac{W}{k} + \frac{W}{k}\left[1 + \left(\frac{2hk}{W}\right)\right]^{\frac{1}{2}} \qquad (29)$$

and the force acting on the structure becomes

$$F = k\delta = W + W\left[1 + \left(\frac{2hk}{W}\right)\right]^{\frac{1}{2}} \qquad (30)$$

Note that if $h = 0$ (the weight is released just above the structure,

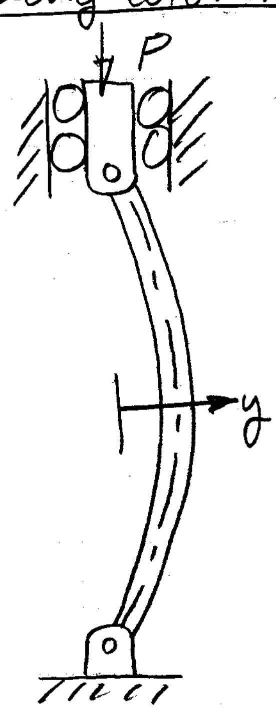
$$F = 2W \qquad (31)$$

Not all systems can be analyzed for impact this easily.

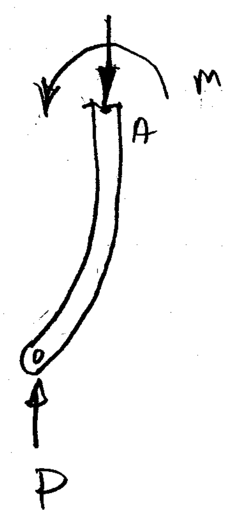
Compression Members

1. Long columns with central loading
2. Intermediate columns with central loading
3. Columns with eccentric loading
4. Short columns with eccentric loading

1. Long Columns



$$\Sigma M_A = M + Py = 0 \qquad (32)$$



From pevious work we can write,

$$\frac{d^2y}{dx^2} = \frac{-P}{EI}\,y \qquad (33)$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,y = 0 \qquad (34)$$

The solution to this simple harmonic motion equation is,

$$y = A \sin \sqrt{\frac{P}{EI}}\, x + B \cos \sqrt{\frac{P}{EI}}\, x \qquad (35)$$

Evaluating this equation for $y = 0$ at $x = 0$ and $x_i = l$,

$A = 0$ (no buckling) $\quad A \neq 0$ (buckling)

$B = 0$

and therefore,

$$\sin \sqrt{\frac{P}{EI}}\, l = 0 \qquad (36)$$

This equation is true when

$$\sqrt{\frac{P}{EI}} = n\pi \text{ for } n = 1, 2, 3, \cdots \qquad (37)$$

For $n = 1$,

$$P_{cr} = \frac{\pi^2 EI}{l^2} \quad \text{(for pinned ends)} \qquad (38)$$

Similar analyses can be extended to other end conditions,

$$P_{cr} = \frac{C\pi^2 EI}{l^2} \qquad (39)$$

where

	C	
End Conditions	Theoretical	Recommended
Fixed - Free	0.25	0.25
Pinned - Pinned	1	1
Fixed - Pinned	2	1.2
Fixed - Fixed	4	1.2

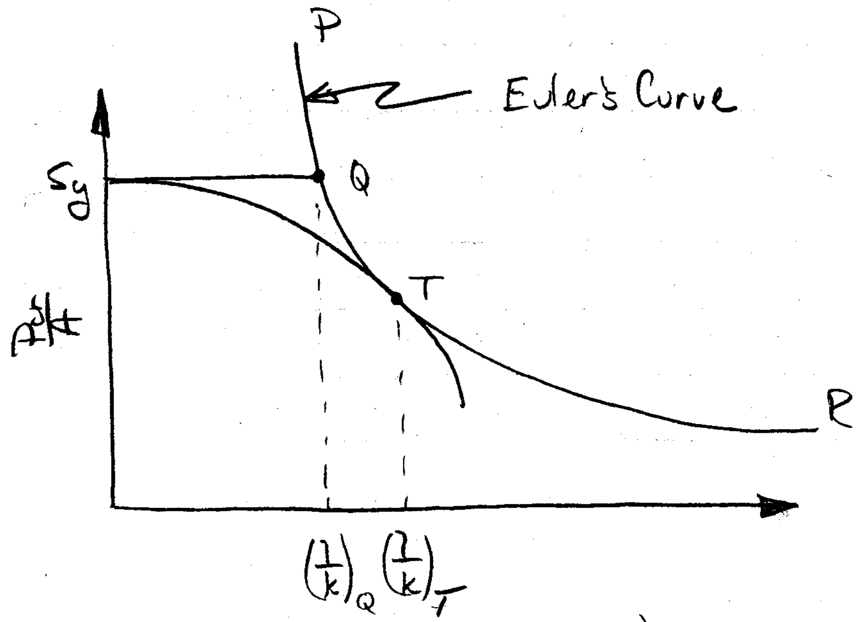
Radius of Gyration

When we allow

$$I = Ak^2 \qquad (40)$$

for the cross-section of the column, k is defined as the radius of gyration. Using the radius of gyration we can define the slenderness ratio as l/k. This ratio can be used as a guide in the application of Euler's equation.

If we look at Euler's curve and its correlation with



Designers usually select point T such that

$$\frac{P_{cr}}{A} = \frac{S_y}{2} \tag{41}$$

and use other metods if the slenderness ratio is less than $\left(\frac{l}{k}\right)_T$

Intermediate-Length Columns w/ Central Loading

Preferred method of designers for $l/k \le (l/k)_1$ is the parabolic formula,

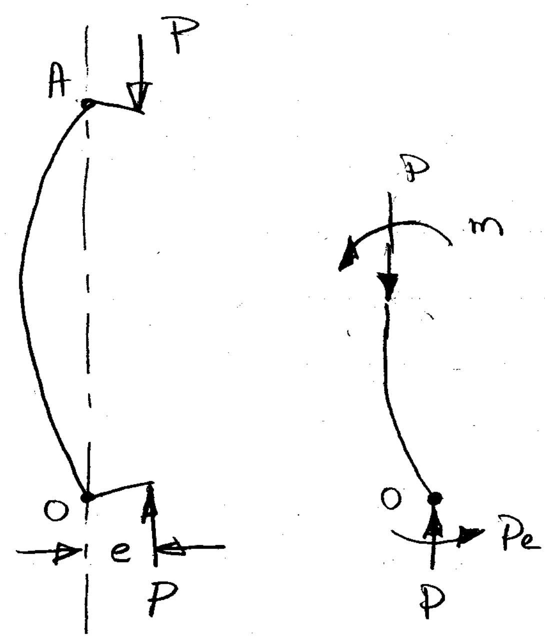
$$\frac{P_{cr}}{A} = a - b\left(\frac{l}{k}\right)^2 \qquad (42)$$

where

$$a = S_y \qquad (43)$$

$$b = \left(\frac{S_y}{2\pi}\right)^2 \frac{1}{CE} \qquad (44)$$

Columns with Eccentric Loads



$$\Sigma M_o = M + Pe + Py = 0 \qquad (45)$$

From previous work we know,

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \qquad (46)$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI} y = - \frac{Pe}{EI} \qquad (47)$$

The initial conditions are

$$x = 0, \quad y = 0$$
$$x = l/2, \quad \frac{dy}{dx} = 0$$

Solving eq. 47 for the initial conditions above we find,

$$\delta = e \left[\sec \left(\frac{l}{2} \sqrt{\frac{P}{EI}} \right) - 1 \right] \qquad (48)$$

and the maximum bending moment is,

$$M_{max} = -P(e + \delta) = -Pe \sec \left(\frac{l}{2} \sqrt{\frac{P}{EI}} \right) \qquad (49)$$

The maximum compressive stress at midspan is

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{P}{A} - \frac{Mc}{Ak^2} \qquad (50)$$

Combining the previous two equations,

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{l}{2k}\sqrt{\frac{P}{EA}}\right)\right] \qquad (51)$$

For design we substitute S_{yc} for σ_c,

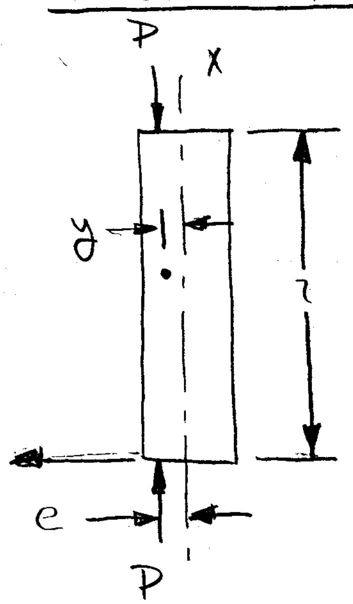
$k \cong$ radius of gyration

$k = \left(\frac{I}{A}\right)^{\frac{1}{2}}$

$$\frac{P}{A} = \frac{S_{yc}}{\left[1 + \frac{ec}{k^2}\sec\left(\frac{l}{2k}\sqrt{\frac{P}{EA}}\right)\right]} \qquad (52)$$

* Note that P occurs on either side of the equation, an iterative solution is required.

Struts and Short Compression Members



Compressive stress in the x-direction is,

$$\sigma_c = \frac{P}{A} + \frac{My}{I} = \frac{P}{A}\left(1 + \frac{ey}{k^2}\right) \qquad (53)$$

The normal stress is zero when

$$y = -\frac{k^2}{e} \qquad (54)$$

The largest compressive stress is,

$$\sigma_c = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right) \qquad (55)$$

When should designers use the "strut" versus "secant" method? If we limit eccentricity from bending to 1%, then the limiting eccentricity is,

$$\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P_{cr}}\right)^{\frac{1}{2}}$$

For slenderness ratios greater than $(l/k)_2$ use the secant method.

Homework Assignment No. 6: Do problems 3-16, 3-28 and 3-50 for next Thursday (Sept. 28).

Criteria for Buckling

1) Long Columns w/ Central Loading (Euler Approach)
2) Intermediate-Length Columns w/ Central Loading (Parabolic Approach)
3) Columns w/ Eccentric Loading
4) Struts of Short Columns w/ Eccentric Loading

l/k slenderness ratio

For l/k ratios $> \left(\frac{2\pi^2 CE}{S_y} \right)^{\frac{1}{2}}$ use Euler Method

For $l/k < \left(\frac{2\pi^2 CE}{S_y} \right)^{\frac{1}{2}}$ use Parabolic Approach